Software and instrumental complex for decision-making on environmental protection from technogenic factors

Cite as: AIP Conference Proceedings **2432**, 030064 (2022); https://doi.org/10.1063/5.0090924
Published Online: 16 June 2022

A. X. Rakhmatullayev, D. A. Saparbayeva and I. E. Tursunov

Software and Instrumental Complex for Decision-making on Environmental Protection from Technogenic Factors

A.X.Rakhmatullayev[1, a)], D.A.Saparbayeva[1], I.E.Tursunov[1, 2]

[1]*Tashkent Institute of Irrigation and Agricultural Mechanization Engineers, 39, Kari Niyaziy str., 100000, Tashkent, Uzbekistan*
[2]*Tashkent Institute of Textile and Light Industry, Tashkent, Uzbekistan*

[a)]*Corresponding author: ravshanzade-09@mail.ru*

Abstract. It is proved that the subspace of the space of the set of all probability measures with more than n supports is a topological manifold modeled by $R^n -$ spaces and is homotopically dense in it.

Keywords: compact, space, probable measure, simplex, manifold, finite dimension, infinite dimension, homeomorphism, homotopy.

INTRODUCTION

In this note, consider a compact set *X*, its space *P*(*X*) of probability measures. In what follows, we study the properties of this space and its subspaces.

METHOD

Using the arbitrariness of the points of the considered space for the interior points of $P_{n,n-1}(X)$, we find a homeomorphic space. The homotopy is used to prove that the above space is homotopy in $P_{n,n-1}(X)$, and the homotopy required for the proof is constructed.

In this note, we prove the subspaces $P_{n,n-1}(X)$ of all probability measures $P(X)$, whose supports consist of exactly n points; it is an $(n-1)-$dimensional topological manifold.

For compact sets X, there is a simple topological classification of the spaces $P(X)$ of all probability measures. In the case of a finite $n-$point space $X=\{n\}$, the points μ of the space $P(n)=P_n(n)$ are convex linear combinations of Dirac measures:

$$\mu = m_0\delta(0)+m_1\delta(1)+\ldots+m_{n-1}\delta(n-1)$$

Therefore, they are naturally identified with points of the $(n-1)-$dimensional simplex σ^{n-1}. Moreover, the Dirac measures $\delta(i)$ form the vertices of the simplex, a the masses m_i placed at the points i are the barycentric coordinates of the measure μ. Thus, $P(n)$ is affinely homeomorphic to the simplex σ^{n-1} [1].

In the case of an infinite compactum X, the space $P(X)$ is also compact (P preserves weight). Further, it, containing simplices of an arbitrarily large number of dimensions, is infinite-dimensional. By Cayley's theorem [2]. a convex compact set $P(X) \subset R^{C(X)}$ is affinely embedded in ℓ_2. Therefore, by Keller's theorem, the compact set $P(X)$, as an infinite-dimensional convex compact set lying in ℓ_2, is homeomorphic to the Hilbert cube $Q = I^{\chi_0}$. On the other hand, the space $P(X)$ of all probability measures on a compact set X is called the set of all regular Borel probability measures on X, equipped with the weakest of the topologies for which each functional $f_u : C(X) \to R$ is continuous, taking the measure μ to $\mu(U)$ (U is an open set in X).

For an arbitrary compact set X and measure $\mu \in P(X)$, its support $\operatorname{supp}(\mu)$ is defined; this is the smallest of the closed sets $F \subset X$ for which $\mu(F) = \mu(X)$, i.e. $\operatorname{supp}(\mu) = \bigcap\{A : A = \overline{A},\ \mu \in P(A)\}$;

$P_n(X) = \{\mu \in P(X) : |\operatorname{supp}\mu| \le n\} -$ is the set of all measures M with at most N supports.

Definition [2]. A topological space X is called a manifold modeled on the space Y, or a $Y-$manifold if every point of the space X has a neighborhood homeomorphic to an open subset of the space Y.

Definition [2]. A subset $A \subset X$ of a space X is called homotopy dense in X if there exists a homotopy $h(x,t) : X \times [0,1] \to X$ such that $h(x,0) = id_X$ and $h(x,(0,1]) \subset A$.

For an infinite (any) compact set X and any $n \in N$ functor P_n, we put $P_{n,n-1}(X) = P_n(X) \setminus P_{n-1}(X)$.

Theorem 1. For any zero-dimensional compactum X, the subspace $P_{n,n-1}(X)$ of the space $P_n(X)$ is an $(n-1)-$dimensional manifold and is homotopy dense in $P_n(X)$.

Proof. Let X be an arbitrary compact. Two cases are possible.

1^0. X is a finite set. For definiteness, let X consist of n points. Then $P_n(X) = P_n(\tilde{n}) = \sigma^{n-1}$ is an $(n-1)-$dimensional standard simplex. $T(x_1, x_2, ..., x_n) = \sigma^{n-1}$ is a simplex from the vertex at points x_i. And the subspace $P_{n,n-1}(\tilde{n}) = T(x_1, x_2, ..., x_n) \setminus F_r(T(x_1, x_2, ..., x_n) = \operatorname{int} T(x_1, x_2, ..., x_n)$, i.e. $P_{n,n-1}(\overline{n}) = T^0(x_1, x_2, ..., x_n) -$ interior of a simplex. The interior is $T^0(x_1, x_2, ..., x_n) -$homeomorphic to the space R^{n-1} [3], that is, $P_{n,n-1}(\overline{n})$ is an $(n-1)$-dimensional manifold R^{n-1}.

2^0. X is an infinite compact. It is known that the space $P_n(X)$ consists of a linear combination of Dirac measures of the form:

$$\mu = m_1\delta_1 + m_2\delta_2 + ... + m_n\delta_n \tag{1}$$

where $\delta_{x_i} -$ is the Dirac measure, $x_i \in X$, $0 \le m_i \le 1$, $\sum_{i=1}^{n} m_i = 1$.

Consider the subspace $P_{n,n-1}(X)$ of the space $P_n(X)$. Obviously $P_{n,n-1}(X)$ is open in $P_n(X)$. Take an arbitrary point $\mu \in P_{n,n-1}(X)$, then

$$\mu = m_1\delta_1 + m_2\delta_2 + ... + m_n\delta_n,$$

$x_1, x_2, \ldots x_n -$ are mutually different, i.e. $|x_1, x_2, \ldots x_n| = n$ and $m_i > 0$, $m_i < 1$. Hence, $\mu \in T^0(x_1, x_2, \ldots, x_n) = \operatorname{int} T(x_1, \ldots, x_n) -$ is the interior of the simplex. It is known that $T^0(x_1, x_2, \ldots, x_n)$ is homeomorphic to the space R^{n-1}. As $O(\mu)$, we identify the set $T^0(x_1, x_2, \ldots, x_n)$, that is, each point of $P_{n,n-1}(X)$ has a neighborhood homeomorphic to R^{n-1}. Hence, $P_{n,n-1}(X)$ is an $(n-1)$-dimensional manifold.

Now we show that the subspace $P_{n,n-1}(X)$ is homotopy dense in $P_n(X)$. We construct the desired homotopy $h(\mu,t): P_n(X) \times [0,1] \to P_n(X)$ by a shallow $h(\mu,t) = (1-t)\mu + t \cdot r(\mu)$, where $t \in [0,1]$, $\mu \in P_n(X)$, and $r(\mu): P(X) \to P(\operatorname{supp} \mu)$ are barycentrically open mapping [3]. If, $t = 0$ then $h(\mu,0) = (1-0)\mu + 0 \cdot r(\mu) = \mu$ i.e. $h(\mu,0) = id_{P_n(X)}$.

If $t \in (0,1]$, then $h(\mu,t) = (1-t)\mu + t \cdot r(\mu) \in P_n(X)$ $\operatorname{supp} h(\mu,t)$ consists of exactly $n-$ distinct points. those. $h(\mu,t) \in P_{n,n-1}(X)$. This means that $P_{n,n-1}(X)$ is homotopy dense in $P_n(X)$. The theorem is proved.

If X is an infinite compact set, then X has a countable proper everywhere dense subset, i.e. $|A_0| = \chi_0$ and $\overline{A} = X$.

Let A be of the form $\{x_1, x_2, \ldots, x_n, \ldots\}$ i.e. $A = \{x_i : i \in N, x_i \in X\}$.

For any $n \in N$, put

$$A_n = \{x_i : x_i \in A, \, i = \overline{1,n}\}$$

In this case, there is the following chain of subspaces A_i for which takes place:

a) $A_i = \{\text{point}\}$;

b) A_n consists of n points of X. that is, $|A_n| = n$;

c) $A_1 \subset A \subset \ldots \subset A_n \subset \ldots$;

d) $\overline{A}_n = A_n$, i.e. A_n is closed and compact:

e) $\bigcup\limits_{n=1}^{\infty} A_n = A$.

From the properties of everywhere dense subsets of infinite compact sets and the properties of the functor P of probability measures, the subspace $P(A)$ is everywhere dense in $P(X)$ and $P(A) = P(\bigcup\limits_{n=1}^{\infty} A_n) = \bigcup\limits_{n=1}^{\infty} P(A_n)$.

On the other hand, the set $P_n(A)$ is also everywhere dense in $P_n(X)$ and $P_n(A) = P_n(\bigcup\limits_{k=1}^{\infty} A_k) = \bigcup\limits_{k=1}^{n} P_n(A_k)$. Consider the set

$$P_n(X) \setminus P_n(A) = P_n(X) \setminus \bigcup_{k=1}^{\infty} P_n(A_k).$$

For a functor P_n, a compact set X from a closed subset $A \subset X$ has equality.

$$P_n(X \setminus A) = P_n(X) \setminus S_{P_n}(A) \qquad (1)$$

Here we denote by $S_{P_n}(A)$ the set $\{\mu \in P_n(X) : \operatorname{supp}_{P_n}(\mu) \cap A \neq \varnothing\}$.

Theorem 2. For any compact set X and any closed subset $A \subset X$ other than X, the subspace $S_{P_n}(A)$ is homotopy dense in $P_n(X)$.

Proof. Let X be an arbitrary compact and $A \subset X$, A closed in X, $A \neq X$.
Two cases are possible:
1. The compact set X is finite;
2. The compact X is infinite;
1. If X is a finite $n-$elementary set, then $P_n(X)$ is affinely homeomorphic to the simplex σ^{n-1}.

In this case, the set $S_{P_n}(A)$ is not empty and convex. Therefore, is homotopy dense in $S_{P_n}(A)$. 2. Let X be infinite and $A \neq X$. We construct the desired homotopy $h(\mu,t) : P_n(X) \times [0,1] \to P_n(X)$ by sloping $h(\mu,t) = (1-t)\mu + t\mu_0$ where μ_0 is a fixed point of $P_n(X \setminus A)$. For example, the Deere measure δ_{x_0} at the point $x_0 \in X \setminus A$, $t \in [0,1]$.

If $t = 0$, then $h(\mu,0) = (1-0)\mu + 0 \cdot \mu_0 = \mu$. i.e. $h(\mu,0) = id_{P_n(X)}$.

If $t \in (0,1)$, then $h(\mu,t) = (1-t)\mu + t \cdot \mu_0$. In this case, the support of the measure $h(\mu,t)$ contains points $\operatorname{supp}\mu$ and points $\operatorname{supp}\mu$, that is, $\operatorname{supp} h(\mu,t) \supseteq \operatorname{supp}\mu \cup \operatorname{supp}\mu_0$. This means that $h(\mu,t) \in S_{P_n}(A)$.

Consequently, the set $S_{P_n}(A)$ is homotopy dense in $P_n(X)$. Theorem 2 is proved.

Theorem 3. For any infinite compactum X and its closed subset A different from X, the subspace $P(X) \setminus P(A)$ is homotopy dense in $P(X)$.

Proof. Let X be an infinite compactum and $A \subset X$, $A = \overline{A}$, $A \neq X$. In this case, $P(X)$ is homeomorphic to the Hilbert cube $Q = \prod_{i=1}^{\infty}[-1,1]_i$. that is, $P(X) \cong Q = \prod_{i=1}^{\infty}[-1,1]_i$, where $[-1,1]$ is a segment in R.

We construct the desired homotopy $h(\mu,t) : P(X) \times [0,1] \to P(X)$ by sloping

$$h(\mu,t) = (1-t)\mu + t\mu_0$$

where $\mu \in P(X)$, $\mu_0 \in P(X) \setminus P(A)$.

If $\mu \in P(X)$ and $t = 0$, then $h(\mu,t) = (1-0)\mu + 0 \cdot \mu_0 = \mu$. those. $h(\mu,0) = id_{P(X)}$.

If $t \in (0,1]$, then $h(\mu,t) = (1-t)\mu + t\mu_0$. The support of the measure $h(\mu,t)$ contains not entirely in the set A. Consequently, the measure $h(\mu,t) \overline{\in} P(A)$. Hence, the measure $h(\mu,t) \in P(X) \setminus P(A)$. This means that the subspace $P(X) \setminus P(A)$ is homotopy dense in $P(X)$. Theorem 3 is proved.

CONCLUSION

For any compact set X and any point $x_0 \in X$, it is true:

a) $S_{P_n}(x_0)$ is homotopy dense in $P_n(X)$;

b) the subspace $P(X) \setminus \delta_{x_0}$ is homotopy dense in $P(X)$.

REFERENCES

1. T. F. Zhuraev and Z. O. Tursunova, O Some geometric and topological properties of probabilistic measures, defined in the infinite compact set of the *Uzbek Mathematical Journal,* 2016, No. 1b pp. 39-48.
2. T. Banakh and T. Radul, Zarichny M. Absorbing sets in infinite - dimensional Manifolds, Math Studies Monogh. Ser. V.1. *VNTL Publishers,* 1996.
3. V.V. Fedorchuk, Probabilistic measures in topology *Uspekh.Mat.Nauk,* 1991, Vol. 46, no. 1 (277), pp. 41-80.